Matchings under Preferences: Strength of Stability and Trade-Offs

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University of Warsaw

Stable Matchings

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Each agent has preferences over the agents in the other set.

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1:	a	b	c	a:	2	1	3
2:	b	c	a	b:	3	2	1
3:	c	a	b	c:	1	3	2

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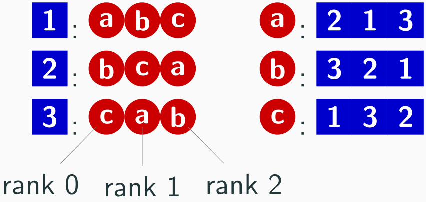

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(rank 0, rank 1, rank 2)

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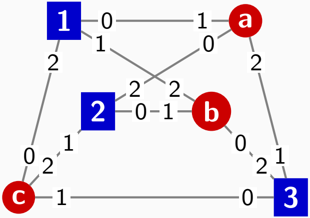

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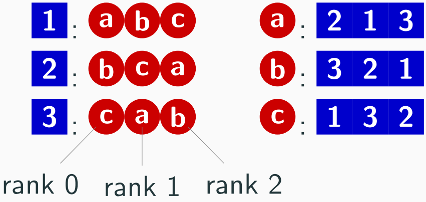

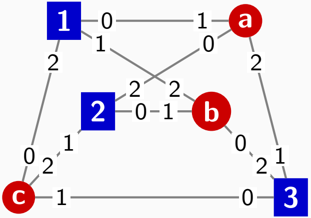

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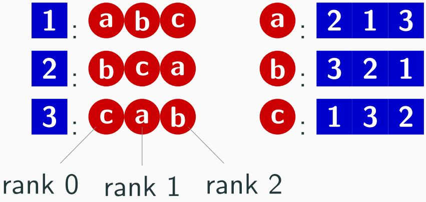

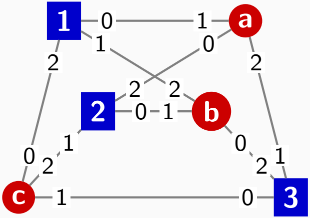

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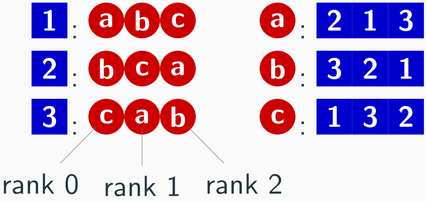

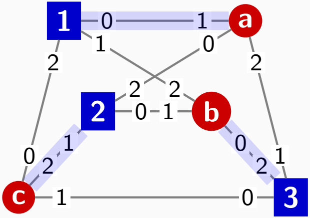

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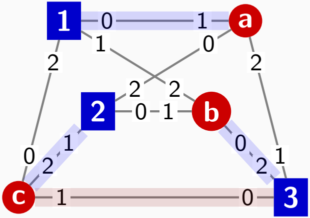

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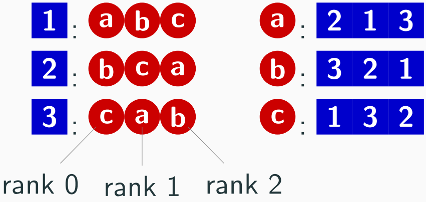

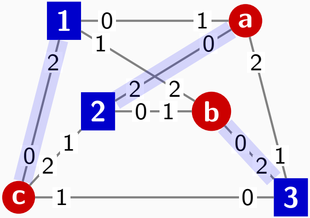

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Stability: Qualitative vs. Quantitative

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Robust:

Agents may change preferences slightly.

Matching shall remain stable.

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Matching stable

Nearly stable:

There are blocking pairs.

But low incentive to switch.

Robustness

Scenario 1: Agents' preferences are not quite accurate.

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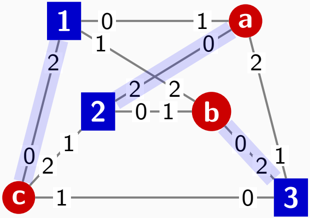
Definition (Robust Matching)

Let $d \in \mathbb{N}$. A matching M for a profile P is d-robust if M is stable in each profile Q that is at most d swaps away from P.

Robustness: Example



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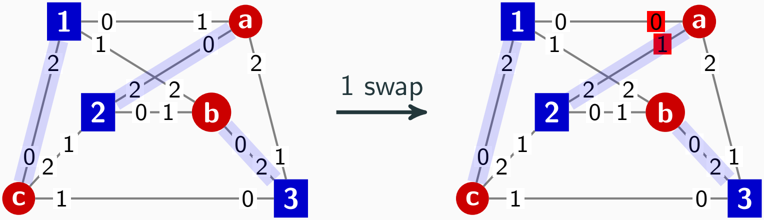

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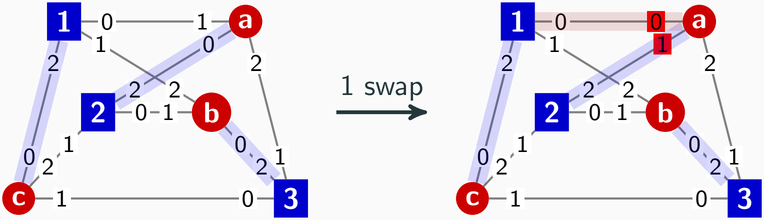

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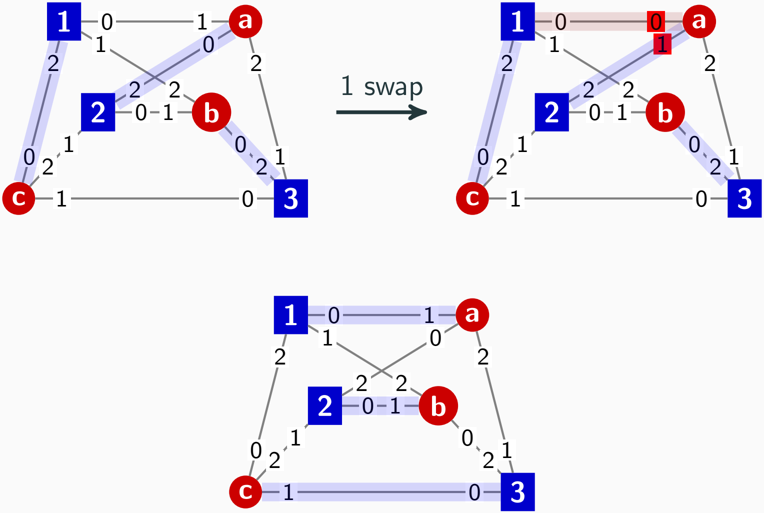

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A matching M for a profile P is d-robust if M is stable in each profile Q that is at most d swaps away from P.

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Proof outline

- $O(n^d)$ profiles $\rightsquigarrow$ $O(n^4)$ critical profiles
- Modify rotation poset: add info from critical profiles
- Find closed subset of rotations in modified rotation poset

Critical Profiles

Fix matching M that is not d-robust.

Profile P —($\leq d$ swaps)→ Profile Q

M stable → M unstable

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Fix matching M that is not d-robust.

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Blocking pair $\{u, v\}$ for M in Q:

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Blocking pair $\{u, v\}$ for M in Q:

u : $\cdots$ $M(u)$ $\cdots$ v $\cdots$ → u : $\cdots$ v $\cdots$ $M(u)$ $\cdots$

v : $\cdots$ $M(v)$ $\cdots$ u $\cdots$ → v : $\cdots$ u $\cdots$ $M(v)$ $\cdots$

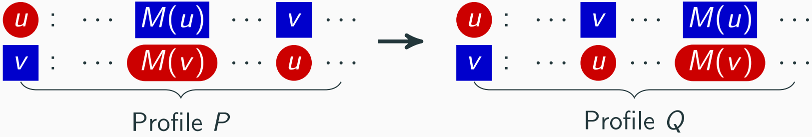
Profile P → Profile Q

Fix matching M that is not d-robust.

Profile P $\xrightarrow{\leq d \text{ swaps}}$ Profile Q

M stable $\qquad\qquad$ M unstable

Blocking pair $\{u, v\}$ for M in Q:



Critical profile R: Move only v, u before $M(u)$, $M(v)$, respectively.

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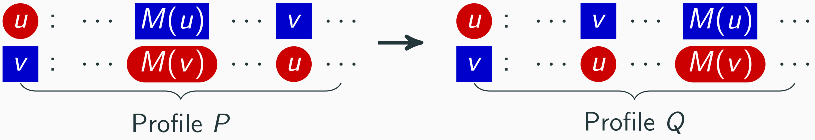

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Observation: M is d-robust iff M is stable in each critical profile.

Critical Profiles

Fix matching M that is not d-robust.

Profile P $\xrightarrow{\leq d \text{ swaps}}$ Profile Q

M stable M unstable

Blocking pair $\{u, v\}$ for M in Q:

Profile P:

u : $\cdots$ $M(u)$ $\cdots$ v $\cdots$

v : $\cdots$ $M(v)$ $\cdots$ u $\cdots$

$\rightarrow$

Profile Q:

u : $\cdots$ v $\cdots$ $M(u)$ $\cdots$

v : $\cdots$ u $\cdots$ $M(v)$ $\cdots$

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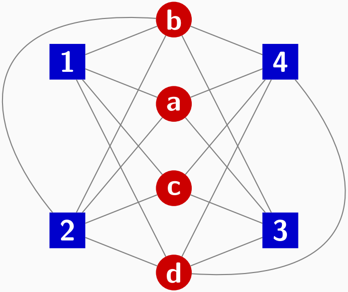
After $O(n^4)$-time preprocessing, testing whether two pairs together in stable matching can be done in $O(1)$ time. [Gusfield, Irving '89].

Primer on Rotations

1 : b c a d
2 : c d b a
3 : d a c b
4 : a b d c

a : 1 2 3 4
b : 2 3 4 1
c : 3 4 1 2
d : 4 1 2 3



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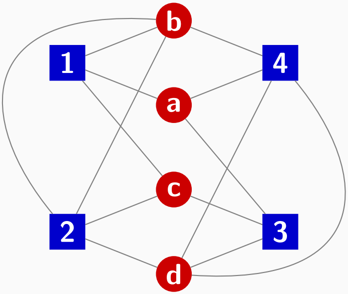

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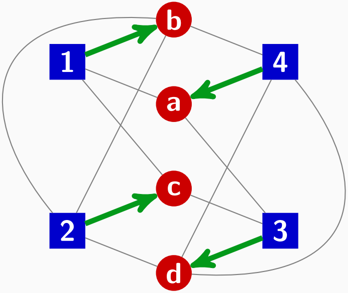

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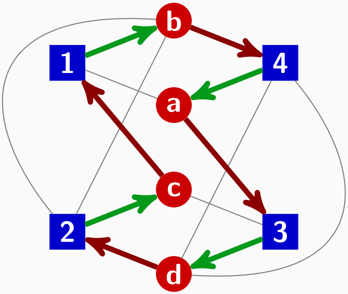

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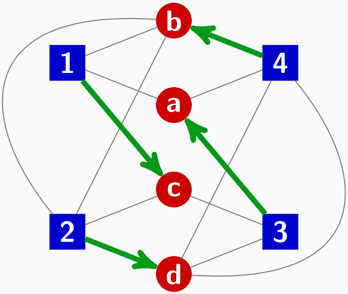

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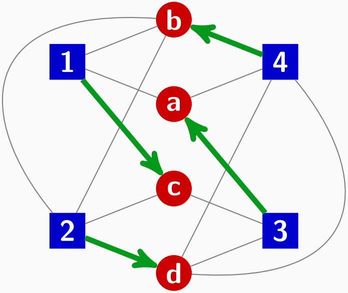

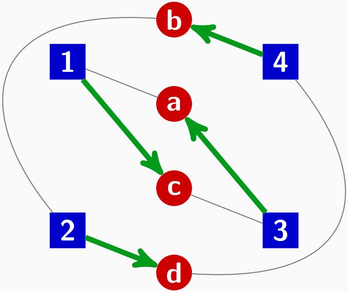

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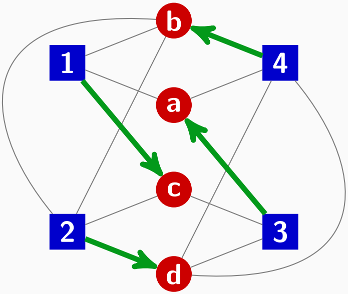

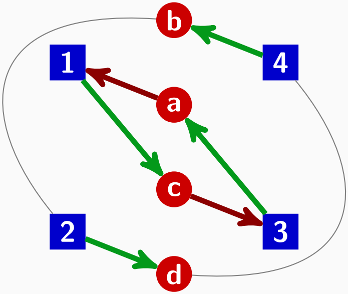

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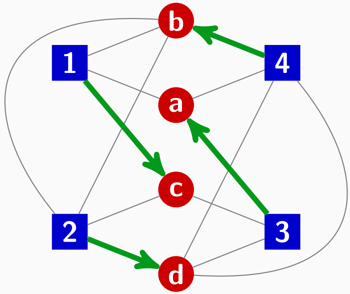

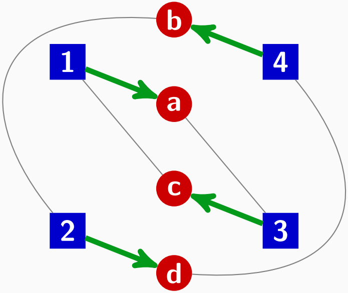

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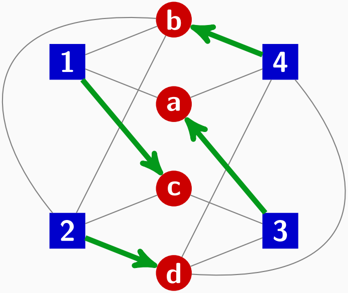

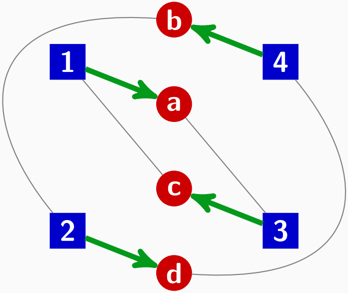

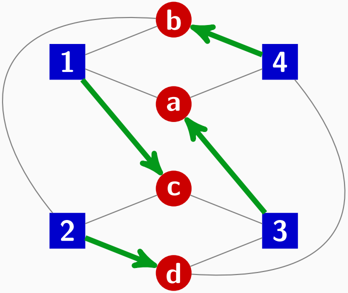

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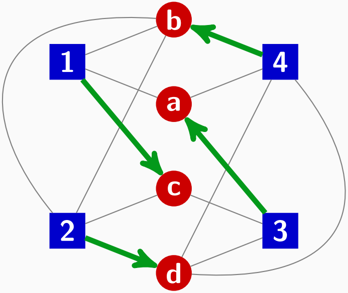

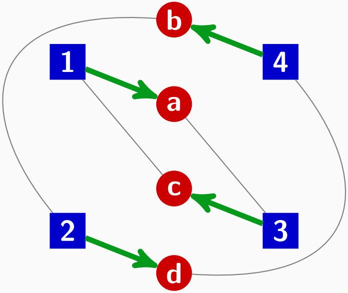

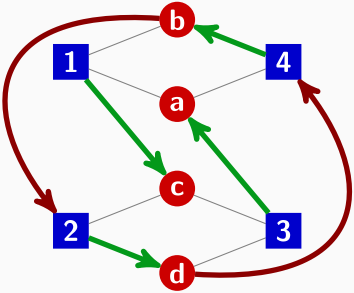

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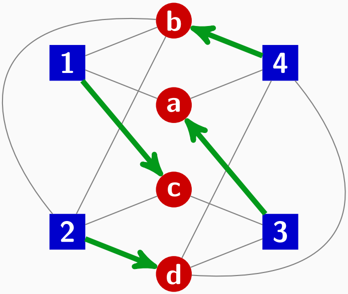

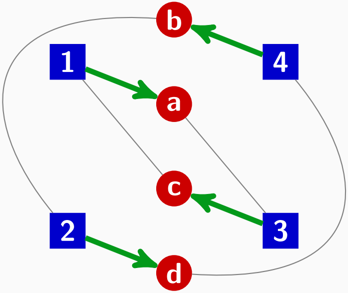

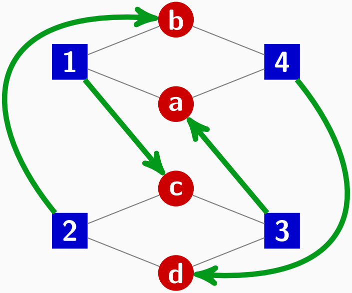

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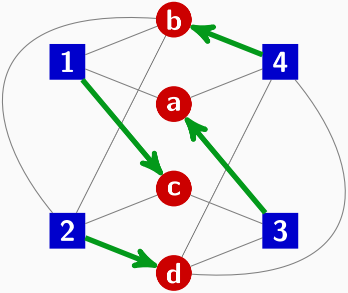
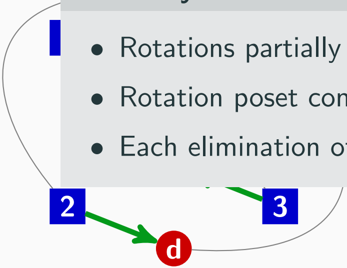
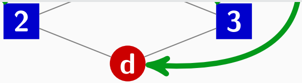


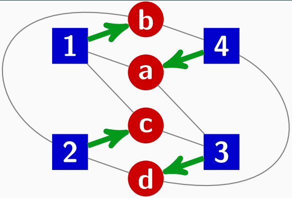
Summary

- Rotations partially ordered by order of exposure
- Rotation poset computable in $O(n^2)$ time [Gusfield, Irving '89]
- Each elimination of rotation: girls improve, boys worsen





Rotation Elimination Graph



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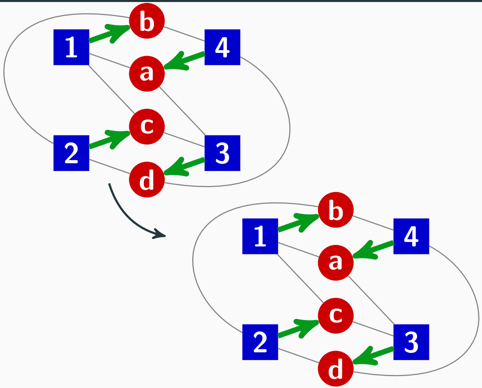

1	:	b	c	a	a	:	1	3	4
2	:	c	d	b	b	:	2	4	1
3	:	d	a	c	c	:	3	1	2
4	:	a	b	d	d	:	4	2	3

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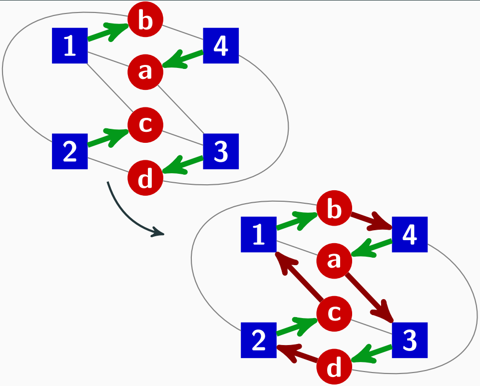

1 :	b	c	a		a :	1	3	4
2 :	c	d	b		b :	2	4	1
3 :	d	a	c		c :	3	1	2
4 :	a	b	d		d :	4	2	3

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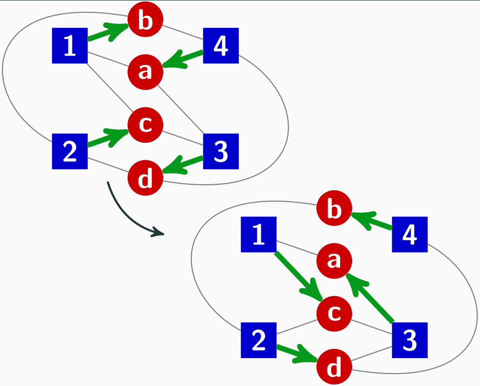

1	:	b	c	a	a	:	1	3	4
2	:	c	d	b	b	:	2	4	1
3	:	d	a	c	c	:	3	1	2
4	:	a	b	d	d	:	4	2	3

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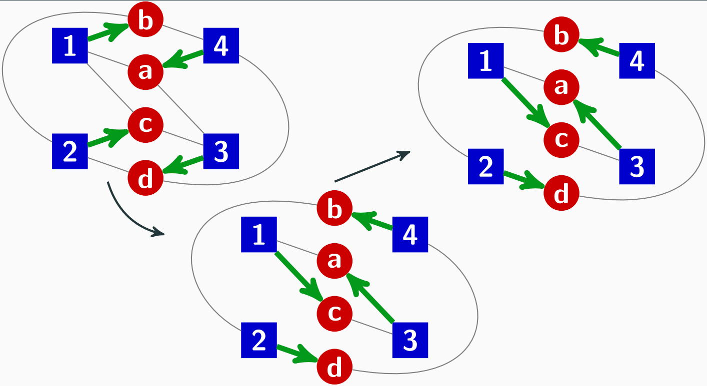

1	:	b	c	a	a	:	1	3	4
2	:	c	d	b	b	:	2	4	1
3	:	d	a	c	c	:	3	1	2
4	:	a	b	d	d	:	4	2	3

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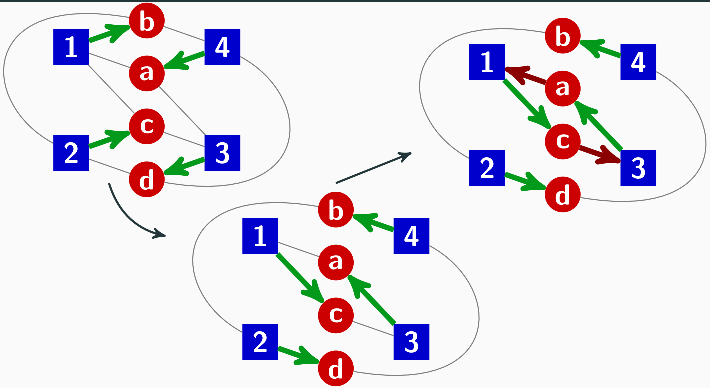

1 :	b	c	a		a :	1	3	4
2 :	c	d	b		b :	2	4	1
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4 :	a	b	d		d :	4	2	3

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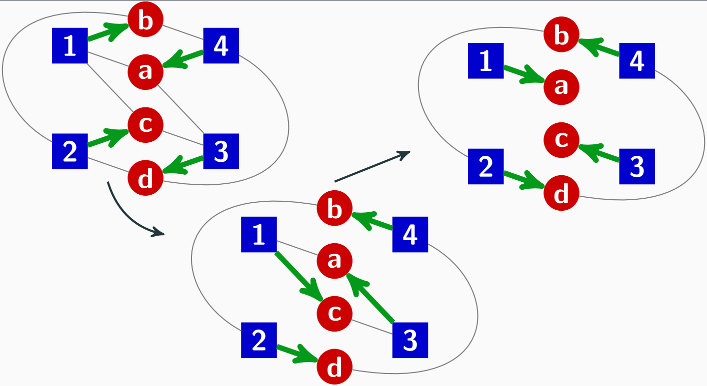

1 :	b	c	a	a :	1	3	4
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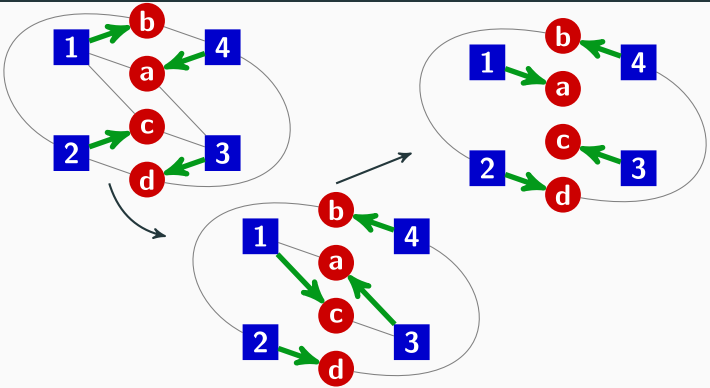

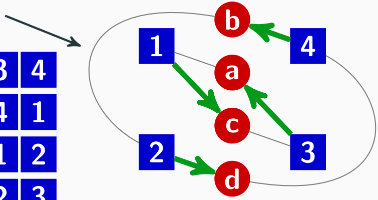

1	:	b	c	a
2	:	c	d	b
3	:	d	a	c
4	:	a	b	d

a	:	1	3	4
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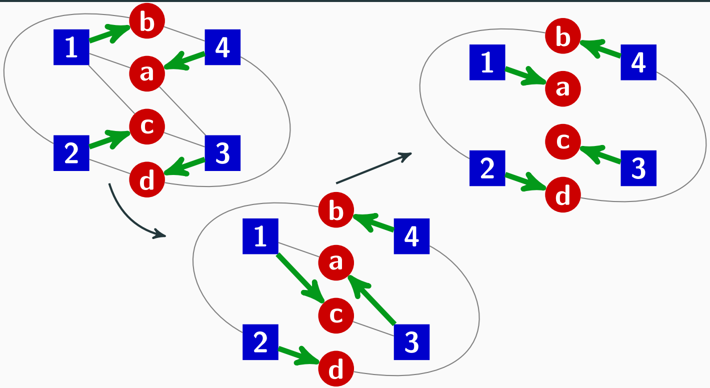
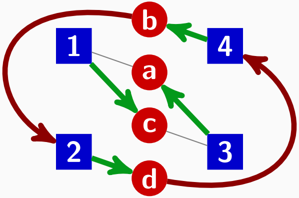
Rotation Elimination Graph





1 :	b	c	a
2 :	c	d	b
3 :	d	a	c
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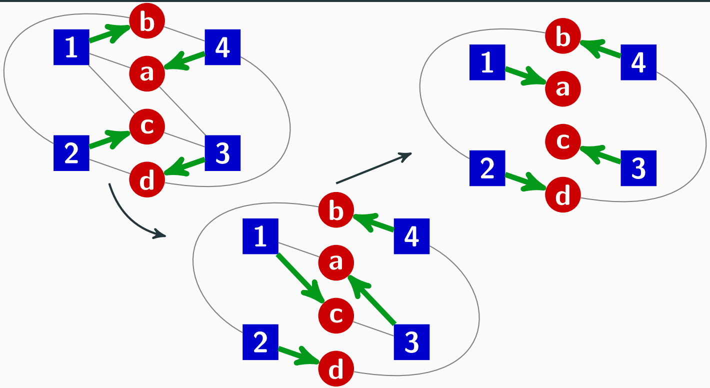

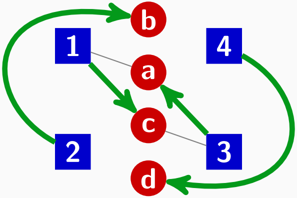

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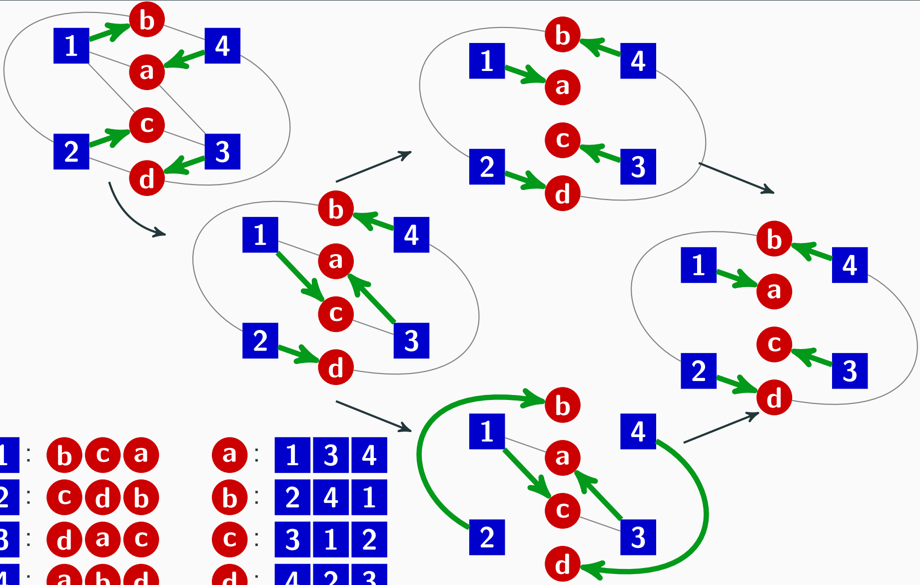

1	: b c a		a	: 1 3 4
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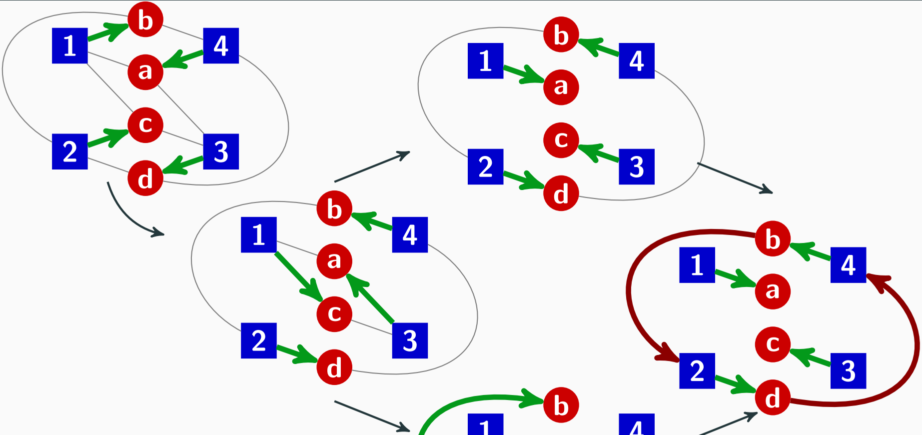

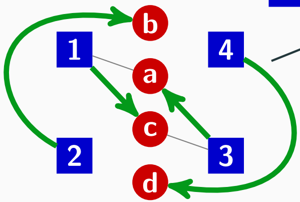

1 :	b	c	a
2 :	c	d	b
3 :	d	a	c
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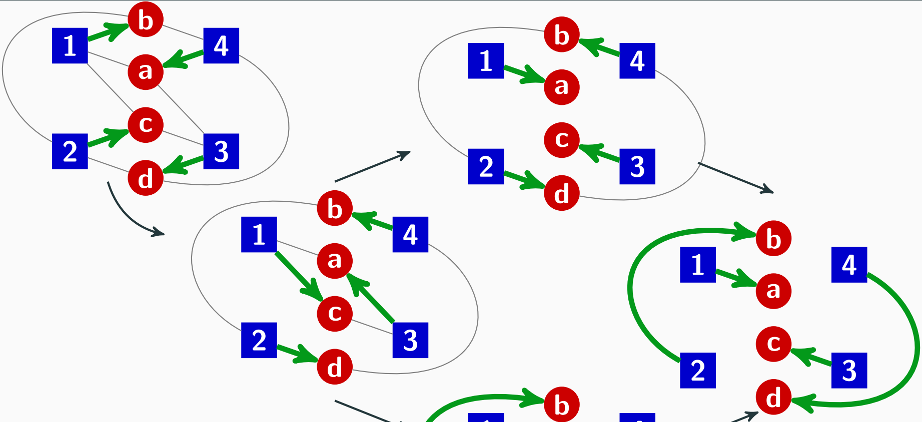

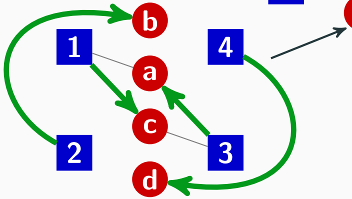

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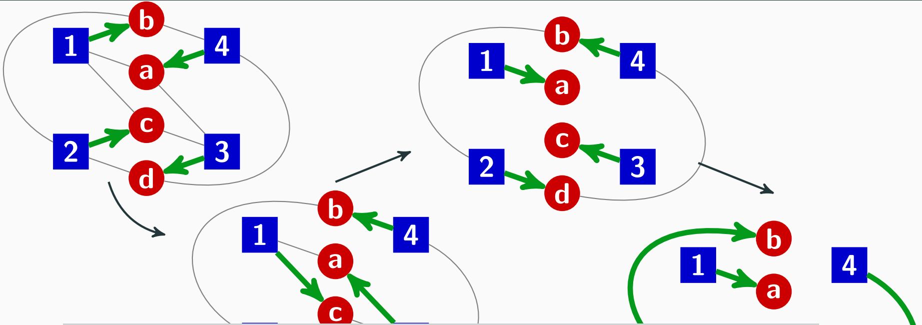
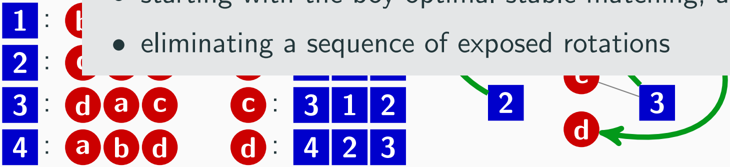


Summary

Each stable matching can be reached by

- starting with the boy-optimal stable matching, and
- eliminating a sequence of exposed rotations



Rotation Poset

1	:	b	c	a	d
2	:	c	d	b	a
3	:	d	a	c	b
4	:	a	b	d	c

a	:	1	2	3	4
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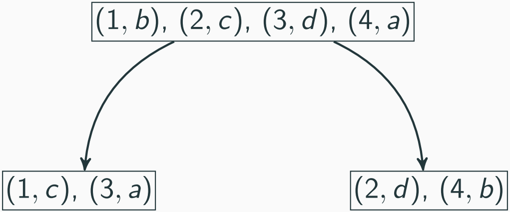

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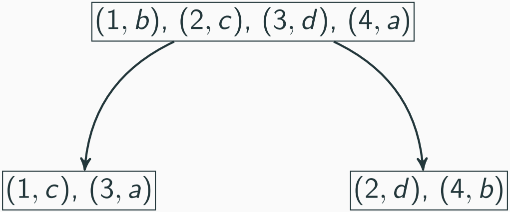

a	:	1	2	3	4
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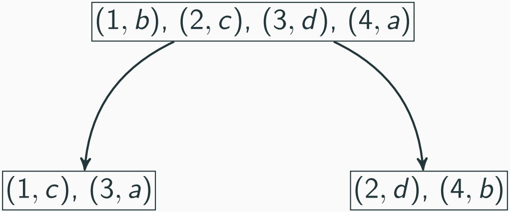

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Closed subsets one-to-one correspond to stable matchings.

Rotations and Action on Critical Profiles

An arbitrary stable matching M can be obtained by

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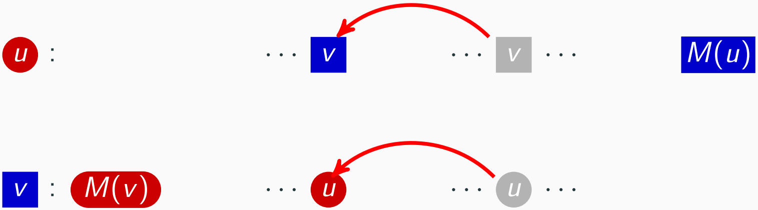
- successively eliminate an exposed rotation.

In critical profile $R(u, v, u^*, v^*)$:



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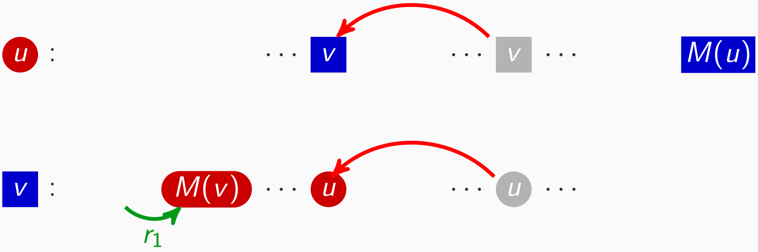

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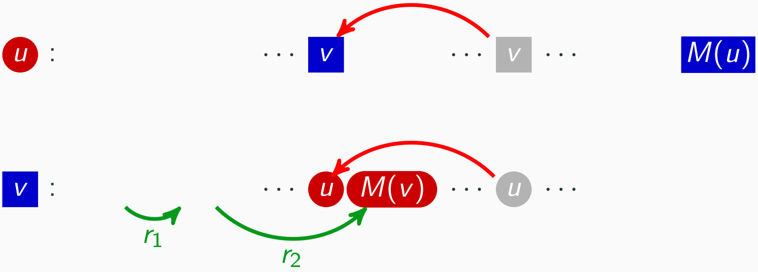

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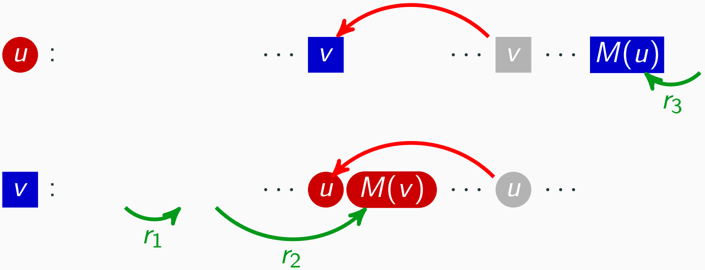

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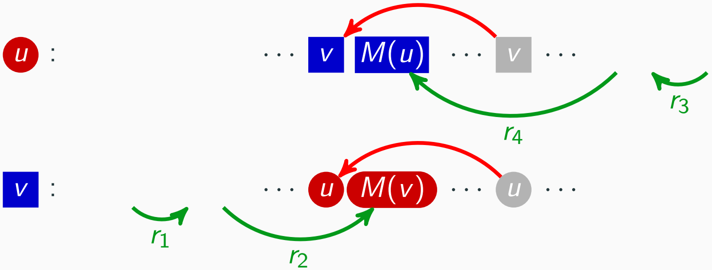

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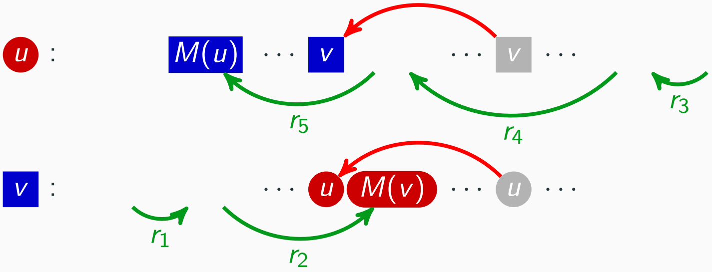

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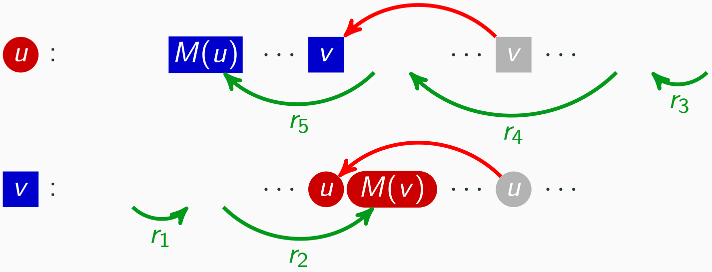
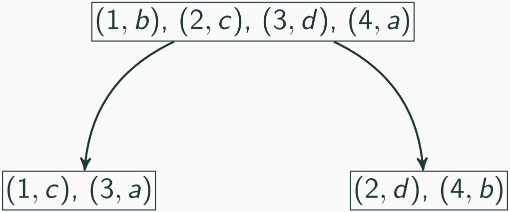


Three types of rotations: Implications, Forbidden, Necessary

Rotation Poset

1 :	b c a d
2 :	c d b a
3 :	d a c b
4 :	a b d c



a :	1 2 3 4
b :	2 3 4 1
c :	3 4 1 2
d :	4 1 2 3

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1 :	b	c	a	d
2 :	c	d	b	a
3 :	d	a	c	b
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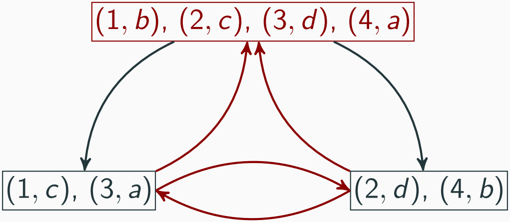

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Near Stability

Scenario 1: Rematches come with a price.

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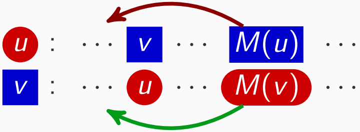

Near Stability

Scenario 2: Tradeoff between distance to stability and secondary objectives.

Definition

Let $d \in \mathbb{N}$. A matching M for a profile P is globally d-nearly stable if making at most d swaps in P makes M stable.

Definition

The egalitarian cost of a matching M is

$$\sum_{\{u,v\} \in M} \mathrm{rank}_u(v) + \mathrm{rank}_v(u).$$

Near Stability: Example

	a_1 :	b_1 b_2		b_1 :	a_1 a_n
$2 \leq i \leq n-1$:	a_i :	b_i b_{i+1}		b_i :	a_{i-1} x_1 $\cdots$ x_n a_i
	a_n :	b_n b_1		b_n :	a_{n-1} x_1 $\cdots$ x_n a_n
$1 \leq i \leq n$:	x_i :	y_i b_1 $\cdots$ b_n		y_i :	x_i

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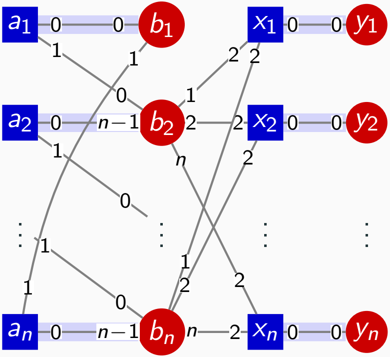

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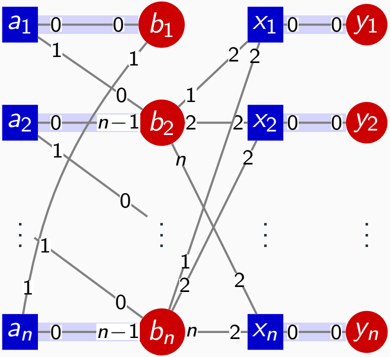

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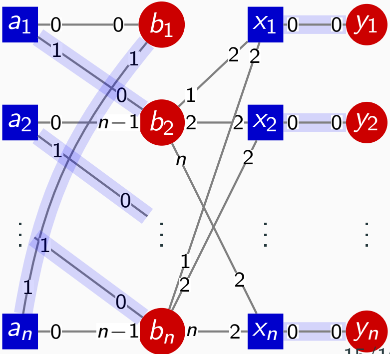
$\rightsquigarrow$



Egal. cost: $n+1$, 1 bp

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Potential follow-ups:

- How robust are boy/girl-optimal matchings in real data?
- Preference restrictions $\rightsquigarrow$ tractable cases for near stability?